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Exactly solvable models for 2D interacting fermions*

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Abstract

I discuss many-body models for correlated fermions in two space dimensions which can be solved exactly using group theory. The simplest example is a model of a quantum Hall system: two-dimensional (2D) fermions in a constant magnetic field and a particular non-local four-point interaction. It is exactly solvable due to a dynamical symmetry corresponding to the Lie algebra $\mathrm{gl}_\infty \oplus \mathrm{gl}_\infty$. There is an algorithm to construct all energy eigenvalues and eigenfunctions of this model. The latter are, in general, many-body states with spatial correlations. The model also has a non-trivial zero temperature phase diagram. I point out that this QH model can be obtained from a more realistic one using a truncation procedure generalizing a similar one leading to mean field theory. Applying this truncation procedure to other 2D fermion models I obtain various simplified models of increasing complexity which generalize mean field theory by taking into account non-trivial correlations but nevertheless are treatable by exact methods.

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1. Introduction

In this paper I present a class of exactly solvable many-body models which describe two-dimensional (2D) interacting fermion systems with spatial correlations. By 'exactly solvable model' I mean that there is an algorithm to construct all eigenvalues and eigenstates of the many-body Hamiltonian explicitly. By 'correlated' I mean that the eigenstates, and in particular the ground state, are not Slater states in general. I introduce these models by discussing a simple example which I call *quantum Hall (QH) model*. This is a model of 2D electrons in a magnetic field and interacting with a particular four-point interaction [1]. I then propose a novel interpretation of this model allowing for natural generalizations and leading to a large class of exactly solvable models of interacting fermions. As I will explain, all these

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models can be obtained from more realistic ones by a truncation procedure which generalizes a similar one leading to mean-field theory. As a motivation, I will discuss two such models which have played an important role in condensed matter physics: the BCS model which solved the problem of superconductivity, and the Hartree–Fock model providing a basis for understanding magnetism in metals. These models can be obtained from a more general one by discarding all but particular kinds of terms in the interactions. The Hartree–Fock model is obtained by keeping only the so-called Hartree and Fock terms, i.e. the terms describing two-body scattering processes where the fermion momenta are left unchanged and are exchanged, respectively; (see the figure in (38)). In 2D there are also mixed scattering terms which are Hartree-like in the x- and Fock-like in the y-component of the momenta and vice versa (see the figure in (56)). I observe that the interaction in the QH model is a sum of such mixed terms, and I propose generalized models where, in addition to Hartree and Fock terms, mixed terms are also included. These models are interesting since they include non-trivial spatial correlations but still can be treated by exact methods.

I only mention in passing some physics motivation (and refer readers interested in more details to a review [2]). There are several fascinating phenomena which have been discovered in complex transition metal oxides in recent years. A prominent example is high temperature superconductivity. 2D models of Hubbard-type are generally believed to account for these phenomena, but these models have been found to be very difficult: despite much work they are still poorly understood. I believe that the simplified models proposed here will be useful in this context.

It is important to note that the exact solution of Hartree–Fock-type models in the thermodynamic limit is equivalent to a mean-field approximation of a more complicated model (I will discuss this in more detail in section 3.5.2), and the same is true for the BCS model[1]. I thus propose that the generalized 2D models I define provide a generalized mean-field theory taking into account non-trivial spatial correlations. The latter is somewhat more complicated but still can be computed by exact methods. To be more specific: standard mean-field theory corresponds to minimizing a functional depending on a small number of variation parameters, whereas the generalized mean-field theory is given by a matrix model. More precisely, the grand canonical partition function of the QH model can be written as a Hermitian matrix integral

$$\mathcal{Z} = \int \mathrm{d}X \, \mathrm{e}^{-L^2 F(X)} \tag{1}$$

with a function F only depending on the eigenvalues of the $L \times L$ Hermitian matrix X, and the thermodynamic limit corresponds to $L \to \infty$. Much is known how to treat such matrix models exactly (see e.g. [3] for an introduction to this field). I believe that it is possible to find useful exact expressions for these partition functions.

The simplified models discussed in this paper are such that eigenstates and eigenvalues can be found exactly using group theory. Models of this kind are standard in nuclear physics [4] but have been used surprisingly little in condensed matter physics. An exact solution of a BCS model [5] found by Richardson a long time ago [6] went unnoticed in condensed matter physics until recently [7]. This model and some recent generalizations [8] are similar to the models proposed here in that all of them can be solved using group theory. In fact, the model solved in [6] is similar to what I call Hartree models in that the corresponding Lie groups are

[1] I believe this is the reason why these models are no longer widely known: there are other, more efficient methods to derive mean-field theory.

finite dimensional (e.g. $SU(2)$), whereas my 'mixed' models are related to infinite-dimensional Lie groups.

The plan of the rest of this paper is as follows. In the following section I discuss the QH model. I point out various remarkable properties of this model and present its solution obtained in [1]. Section 3 contains a discussion of general interacting fermion systems and the truncation procedure leading to simplified models which can be solved exactly. The 2D Hubbard model is a special case where this truncation procedure leads to particular simple models. I end with a short summary in section 4.

An outline of this paper, which can be read as extended abstract, appeared in [9].

2. The quantum Hall model

In this section, I discuss a model of interacting fermions proposed and solved in [1].

2.1. Definition and physical interpretation

I start with the Landau Hamiltonian

$$h_B = (-\mathrm{i}\partial_x + By)^2 + (-\mathrm{i}\partial_y - Bx)^2 \tag{2}$$

describing the motion of a charged particle in the (x, y)-plane with a constant magnetic field of magnitude $B > 0$ perpendicular to this plane. I choose units such that the mass and charge of the electron are $1/2$ and 2, respectively. A microscopic model for a quantum Hall system[2] is given by the many-body Hamiltonian $H = H_0 + V$ where the free part is

$$H_0 = \int_{\mathbb{R}^2} \mathrm{d}^2\mathbf{x}\, \psi^\dagger(\mathbf{x})(h_B - \mu)\psi(\mathbf{x}) \tag{3}$$

with fermion creation- and annihilation operators ψ and $\psi^\dagger$ obeying the usual canonical anticommutation relations, $\{\psi(\mathbf{x}), \psi^\dagger(\mathbf{x}')\} = \delta^2(\mathbf{x} - \mathbf{x}')$, etc; $\mathbf{x} \equiv (x, y)$, and the real parameter μ is a chemical potential. The interaction potential V is of the following four-point type:

$$V = \int \mathrm{d}^2\mathbf{x}_1 \cdots \mathrm{d}^2\mathbf{x}_4\, v(\mathbf{x}_1, \ldots, \mathbf{x}_4) : \psi^\dagger(\mathbf{x}_1)\psi(\mathbf{x}_2)\psi^\dagger(\mathbf{x}_3)\psi(\mathbf{x}_4) : \tag{4}$$

with the dots indicating normal ordering as usual[3]. The interaction vertices v of main physical interest for quantum Hall physics are as follows:

$$v(\mathbf{x}_1, \ldots, \mathbf{x}_4) = \delta^2(\mathbf{x}_1 - \mathbf{x}_2)\delta^2(\mathbf{x}_3 - \mathbf{x}_4)W(\mathbf{x}_1 - \mathbf{x}_3) \tag{5}$$

with W the two-body potential. For example, $W(\mathbf{x}) = \mathrm{const}/|\mathbf{x}|$ corresponds to a 3D Coulomb repulsion, but this choice leads to a model which cannot be analysed without approximations. One might hope to obtain a simpler model by taking a fully local interaction, $W(\mathbf{x}) = g_0\delta^2(\mathbf{x} - \mathbf{y})$ with g_0 a coupling constant, but this model is trivial since the fully local interaction potential vanishes due to the Pauli principle, $V = 0$.

The QH model can be obtained as a deformation of a fully local interaction potential which is such that it becomes non-trivial. In Fourier space this deformed interaction is given by

$$V_{\mathrm{QH}} = \frac{g_0}{(2\pi)^2} \int \mathrm{d}^2\mathbf{k}_1 \cdots \mathrm{d}^2\mathbf{k}_4\, \delta^2(\mathbf{k}_1 - \mathbf{k}_2 + \mathbf{k}_3 - \mathbf{k}_4) \times \mathrm{e}^{-\mathrm{i}\theta(\mathbf{k}_1 \wedge \mathbf{k}_2 + \mathbf{k}_3 \wedge \mathbf{k}_4)} : \hat\psi^\dagger(\mathbf{k}_1)\hat\psi(\mathbf{k}_2)\hat\psi^\dagger(\mathbf{k}_3)\hat\psi(\mathbf{k}_4) : \tag{6}$$

[2] I do *not* write 'quantum Hall *effect* model' since, firstly, such a model should also include impurities and, secondly, I do not know if the interactions I propose below can account for the QHE. Note also that I ignore spin.

[3] $: \psi^\dagger(1)\psi(2)\psi^\dagger(3)\psi(4) := \psi^\dagger(3)\psi^\dagger(1)\psi(2)\psi(4)$.

with the hat indicating Fourier transform, $\mathbf{k} \wedge \mathbf{q} = k_x q_y - k_y q_x$, and θ the deformation parameters (i.e. $\theta = 0$ corresponds to the fully local interaction). At first sight this interaction potential might appear somewhat unusual, but it has several remarkable mathematical properties which I will discuss below. These mathematical properties were my original motivation to consider this model [1]. In the following section I will discuss other interacting fermion models which suggest an alternative physical interpretation. This discussion will naturally lead to various interesting generalizations of this QH model which, as I believe, are promising candidates for understanding 2D correlated fermion in a precise mathematical framework.

2.2. Mathematical properties

The QH model defined above has various interesting mathematical properties. Firstly, the interaction potential in (6) can be obtained from the fully local interaction potential by replacing the pointwise product of fields by the so-called Groenewold–Moyal star product,

$$V_{\mathrm{QH}} = g_0 \int_{\mathbb{R}^2} \mathrm{d}^2\mathbf{x} : \psi^\dagger(\mathbf{x}) \star \psi(\mathbf{x}) \star \psi^\dagger(\mathbf{x}) \star \psi(\mathbf{x}) : \tag{7}$$

with $\star$ the associative product defined such that $x \star y - y \star x = -2\mathrm{i}\theta$ [10]. Indeed, one can deduce that the $\star$-product of two plane waves is $\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}} \star \mathrm{e}^{\mathrm{i}\mathbf{q}\cdot\mathbf{x}} = \mathrm{e}^{\mathrm{i}(\mathbf{k}+\mathbf{q})\mathbf{x}}\,\mathrm{e}^{-\mathrm{i}\theta\mathbf{k}\wedge\mathbf{q}}$, and with that one can compute equation (7) by inserting the Fourier transforms of the fields and obtain (6). Field theories with such $\star$-interactions received much attention in the particle physics literature theory recently (see e.g. [11] for reviews). Secondly, the interaction potential V_{QH} has the remarkable property that it looks the same in Fourier- and position space: computing the inverse Fourier transformation one finds that V_{QH} can be written as in equation (4) with the interaction vertex [12]

$$v(\mathbf{x}_1, \ldots, \mathbf{x}_4) = \frac{g_0}{(2\pi\theta)^2} \delta^2(\mathbf{x}_1 - \mathbf{x}_2 + \mathbf{x}_3 - \mathbf{x}_4)\, \mathrm{e}^{-\mathrm{i}(\mathbf{x}_1\wedge\mathbf{x}_2+\mathbf{x}_3\wedge\mathbf{x}_4)/\theta}. \tag{8}$$

Since the Landau Hamiltonian obviously looks the same in position- and Fourier space as well, the Hamiltonian $H = H_0 + V_{\mathrm{QH}}$ is invariant under the following duality transformation [12]:

$$\begin{aligned} &\psi(\mathbf{x}) \to \tilde{\psi}(\mathbf{x}) = B\hat{\psi}(B\mathbf{x}) \\ &B \to \tilde{B} = B \\ &\theta \to \tilde{\theta} = -1/(\theta B^2) \\ &g_0 \to \tilde{g}_0 = g_0/|B\theta|. \end{aligned} \tag{9}$$

Note that this duality transformation involves Fourier transformation, $\psi(\mathbf{x}) \to \hat{\psi}(\mathbf{k})$, followed by a re-scaling of the Fourier variable to correct for the length dimensions, $\mathbf{k} \to \mathbf{k}/B \equiv \mathbf{x}$ (since B has the dimension [length^{-2}]). Thirdly, as I will discuss in more detail below, this model is exactly solvable for $\theta = \pm B^{-1}$ [1]. It is interesting to note that these exactly solvable cases are mapped onto each other under the duality transformation.

One can actually give the solution of a more general model where h_B in equation (3) is replaced by $(1-a)h_B + ah_{-B}$ with $0 \leqslant a \leqslant 1$. This generalization means that one allows for a confining electric potential $\propto (x^2 + y^2)$ in addition to a magnetic field.

2.3. Integrability

In the following I assume $\theta = 1/B$.

As discussed in most quantum mechanics textbooks (see e.g. section 8.6 in [14]), the eigenfunctions $\phi_{\ell m}(\mathbf{x})$ of the Landau Hamiltonian in equation (2) can be chosen such that they

are labelled by two positive integers ℓ, m with eigenvalues depending only on one of these quantum numbers, $h_B\phi_{\ell m} = e_\ell\phi_{\ell m}$ where $e_\ell = 4B\big(\ell - \frac{1}{2}\big)$ (this is the well-known degeneracy of the Landau problem). The quantum numbers ℓ and m can be interchanged either by complex conjugation, $\phi_{\ell m}(\mathbf{x})^* = \phi_{m\ell}(\mathbf{x})$, or by changing the direction of the magnetic field, $h_{-B}\phi_{\ell m} = e_m\phi_{\ell m}$ (see e.g. appendix A in [1]). Note that changing the sign of the magnetic field is equivalent to a parity transformation.

These eigenfunctions provide a complete orthonormal basis in the one-particle Hilbert space $L^2(\mathbb{R}^2)$, and thus the many-body Hamiltonian H_0 in equation (3) can be diagonalized by expanding the fermion field operators in this Landau basis, $\psi(\mathbf{x}) = \sum_{\ell,m} c_{\ell m}\phi_{\ell m}(\mathbf{x})$ and similarly for $\psi^\dagger(\mathbf{x})$. This yields $H_0 = \sum_{\ell,m}(e_\ell - \mu)c^\dagger_{\ell m}c_{\ell m}$. Since the $\phi_{\ell m}$ are common eigenfunctions of h_B and h_{-B}, one can actually generalize the model by replacing $h_B \to (1-a)h_B + ah_{-B}$ and still obtain a diagonal many-body Hamiltonian,

$$H_0 = \sum_{\ell,m}(E_m + \tilde{E}_\ell)c^\dagger_{\ell m}c_{\ell m} \tag{10}$$

with $\tilde{E}_\ell \equiv (1-a)e_\ell$ and $E_m \equiv ae_m - \mu$. The fermion field operators $c^\dagger_{\ell m}$ and $c_{\ell m}$ obey the canonical commutation relations $\big\{c_{\ell m}, c^\dagger_{\ell' m'}\big\} = \delta_{\ell,\ell'}\delta_{m,m'}$, etc. The model is defined on a fermion Fock space with a vacuum Ω such that $c_{\ell m}\Omega = 0$ for all ℓ, m.

The crucial fact making the QH model 'special' are the following remarkable $\star$-product relations of the Landau eigenfunctions [1]:

$$\phi_{\ell m} \star \phi_{\ell' m'} = \sqrt{\frac{B}{4\pi}}\delta_{m,\ell'}\phi_{\ell m'} \qquad \text{for } \theta = \frac{1}{B}. \tag{11}$$

Using that and other properties of the Landau eigenfunctions, a straightforward computation brings the interaction potential in (7) to the following simple form:

$$V_{\mathrm{QH}} = g\sum : c^\dagger_{m\ell}c_{m\ell'}c^\dagger_{m'\ell'}c_{m'\ell} : \tag{12}$$

where $g = g_0 B/4\pi$.

It is interesting to note that the QH Hamiltonian can be written in terms of the following fermion bilinears:

$$\rho_{mm'} = \sum_\ell c^\dagger_{\ell m}c_{\ell m'}, \qquad \tilde{\rho}_{\ell\ell'} = \sum_m c^\dagger_{\ell m}c_{\ell' m}. \tag{13}$$

This allows us to write

$$H_0 = \sum_m E_m\rho_{mm} + \tilde{E}_m\tilde{\rho}_{mm} \tag{14}$$

and

$$V_{\mathrm{QH}} = g\sum_{\ell,m} : \rho_{m\ell}\rho_{\ell m} := -g\sum_{m,\ell} : \tilde{\rho}_{\ell m}\tilde{\rho}_{m\ell} : . \tag{15}$$

If one regards the operators $\rho_{\ell m}$ as elements of an infinite matrix ρ and similarly for $\tilde{\rho}$, one can write this in the following suggestive matrix notation:

$$H_0 = \mathrm{Tr}(E\rho + \tilde{E}\tilde{\rho}), \qquad V_{\mathrm{QH}} = g : \mathrm{Tr}(\rho^2) := -g : \mathrm{Tr}(\tilde{\rho}^2) : \tag{16}$$

with $E = \mathrm{diag}(E_1, E_2, \ldots)$ and similarly for $\tilde{E}$.

Using the fermion anticommutation relations one can derive the following commutation relations:

$$[\rho_{\ell m}, \rho_{\ell' m'}] = \delta_{m,\ell'}\rho_{\ell m'} - \delta_{m',\ell}\rho_{\ell' m} \tag{17}$$

and similarly for the $\tilde{\rho}$, and $[\rho_{\ell m}, \tilde{\rho}_{\ell' m'}] = 0$. Moreover,

$$\rho_{\ell m}^{\dagger} = \rho_{m\ell} \tag{18}$$

and similarly for the $\tilde{\rho}$. One thus sees that the operators ρ and $\tilde{\rho}$ represent the Lie algebra $\mathrm{gl}_\infty \oplus \mathrm{gl}_\infty$. It is not difficult to check that the interaction potential V_{QH} represents a Casimir element in this Lie algebra (i.e. it commutes with 'everything') and the free Hamiltonian H_0 represents an element in the Cartan subalgebra (i.e. is a sum of mutually commuting terms). However, the Lie algebra $\mathrm{gl}_\infty \oplus \mathrm{gl}_\infty$ does *not* commute with H: one can check the following commutation relations, $[H, \rho_{\ell\ell'}] = (E_\ell - E_{\ell'})\rho_{\ell\ell'}$, and similarly for $\tilde{\rho}$ and $\tilde{E}$. Thus this Lie algebra defines a dynamical symmetry of the QH model. This suggests that the model is integrable. I will show this by presenting an algorithm to construct all eigenstates and eigenvalues of H below.

2.4. Solution

A complete orthonormal basis in the Fock space of the model is given by the states

$$|N\rangle = c_{\ell_1 m_1}^{\dagger} c_{\ell_2 m_2}^{\dagger} \cdots c_{\ell_N m_N}^{\dagger} \Omega \tag{19}$$

with $N = 0, 1, 2, \ldots$ the fermion number (for brevity I suppress the dependence on the other quantum numbers). I find it convenient to introduce a regularization by restricting the quantum numbers to $\ell_j, m_j = 1, 2, \ldots, L$ with a finite cut-off L. This has a natural physical interpretation: one adds a box-like potential confining the QH system to the disc $x^2 + y^2 \leqslant \text{const}\, L^2/B$. This reduces the dynamical symmetry to $\mathrm{gl}_L \oplus \mathrm{gl}_L$, and the fermion Fock space becomes finite dimensional, $\dim = 2^{L^2}$.

Straightforward computations using the canonical anticommutation relations show that the interaction potential acts on these states as follows:

$$:\mathrm{Tr}(\rho^2):|N\rangle = 2 \sum_{1 \leqslant i < j \leqslant N} T_{(ij)}|N\rangle \equiv C_N|N\rangle \tag{20}$$

with $T_{(ij)}$ transpositions interchanging m_i and m_j,

$$T_{(ij)}|N\rangle = c_{\ell_1 m_1}^{\dagger} \cdots c_{\ell_i m_j}^{\dagger} \cdots c_{\ell_j m_i}^{\dagger} \cdots c_{\ell_N m_N}^{\dagger} \Omega \qquad i < j. \tag{21}$$

Note that T defines a representation of the permutation group S_N, and $C_N = \sum_{i<j} T_{(ij)}$ commutes with all permutations T_P. All $|N\rangle$ are, of course, eigenstates of the free part of the QH Hamiltonian, $H_0|N\rangle = \mathcal{E}_0|N\rangle$ with $\mathcal{E}_0 = \sum_{j=1}^{N} \left(\tilde{E}_{\ell_j} + E_{m_j}\right)$, and due to the special form of $\mathcal{E}_0$ this free part commutes with all permutations T_P. Thus the eigenstates of H can be constructed by solving the following eigenvalue equation:

$$C_N \sum_{P \in S_N} a_P T_P = \gamma \sum_{P \in S_N} a_P T_P. \tag{22}$$

Each solution of (22) provides and eigenstate ψ and corresponding eigenvalue $\mathcal{E}$ of the QH Hamiltonian H as follows:

$$\psi = \sum_P a_P T_P |N\rangle \qquad \text{and} \qquad \mathcal{E} = \sum_{j=1}^{N} \left(E_{\ell_j} + \tilde{E}_{m_j} y\right) + 2g\gamma. \tag{23}$$

Equation (22) can be solved by constructing and diagonalizing $N! \times N!$ matrices representing C_N. To appreciate the difficulty of this problem it is instructive to do this for $N = 2, 3$. Obviously such brute-force approach is restricted to rather small N. Fortunately for us, equation (22) corresponds to a classical problem in group theory which was solved a long

time ago (the group theory results used in the following are discussed in most advanced group theory books; for more details and specific references see [1]): the representation T is (essentially) the regular representation, and solving equation (22) amounts to decomposing T into irreps. The irreps of S_N are well known. They are one-to-one correspondence with partitions $[\lambda] = [\lambda_1, \lambda_2, \ldots, \lambda_K]$ of N where the λ_j are integers summing up to N and obeying $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_K \geqslant 1$. The possible eigenvalues γ in equation (22) are given by the value of the class operator C_N in the irreps $[\lambda]$,

$$\gamma_{[\lambda]} = \sum_{j=1}^{K} \frac{1}{2}\lambda_j(\lambda_j + 1) - j\lambda_j \tag{24}$$

and its multiplicity is the square of the dimension of the irreps $[\lambda]$. For example, for $N = 4$ the eigenvalues γ of C_4 are $6(1), 2(9), 0(4), -2(9), -6(1)$ with the numbers in the parenthesis giving the multiplicities. The corresponding $\sum_P a_P T_P$ is equal to the Young operator $\mathcal{Y}_{[\lambda]}$ associated with $[\lambda]$. This gives the following explicit recipe for constructing the eigenstates associated with $[\lambda]$: for sets of $J \leqslant N$ distinct integers $(i_1, i_2, \ldots, i_J)$, $1 \leqslant i_j \leqslant N$, define (anti-)symmetrizers $\mathcal{S}^{\pm}_{i_1,i_2,\ldots,i_J} = \sum_{P\in S_J} (\pm 1)^{|P|} T_P^{i_1,i_2,\ldots,i_J}$ acting on states $|N\rangle$ as described above, with the integers i_j indicating the positions of the m_{i_j} on which the permutations act. Write the integers $1, 2, \ldots, N$ into the boxes of the Young tableau associated with $[\lambda]$, take for each row one symmetrizer $\mathcal{S}^+$ with indices given by the numbers in the boxes of the row, and similarly for each column take an antisymmetrizer $\mathcal{S}^-$. Then

$$\psi = \mathcal{Y}_{[\lambda]}|N\rangle \equiv \prod \mathcal{S}^+_{\text{rows}} \prod \mathcal{S}^-_{\text{columns}}|N\rangle. \tag{25}$$

For example, the ten-particle eigenstates associated with $[\lambda] = [5, 3, 2]$ are

$$\psi = \mathcal{S}^+_{1,2,3,4,5}\mathcal{S}^+_{6,7,8}\mathcal{S}^+_{9,10}\mathcal{S}^-_{1,6,9}\mathcal{S}^-_{2,7,10}\mathcal{S}^-_{3,8}|N = 10\rangle \tag{26}$$

and $\mathcal{E} = \sum_{j=1}^{10} \left(E_{\ell_j} + \tilde{E}_{k_j}\right) + 14g$ are the corresponding eigenvalues. This gives all possible eigenvalues and corresponding eigenstates of the QH model. It is worth mentioning that there is another representation $\tilde{T}$ of the permutation group S_N acting on the indices ℓ_j of the states $|N\rangle$ in (19). The two representations are, however, related: interchanging $m_i \leftrightarrow m_j$ and $\ell_i \leftrightarrow \ell_j$ together is the same as interchanging two fermions and only gives a minus sign, and thus $T_P\tilde{T}_P = (-1)^{|P|}$ for all $P \in S_N$.

It is interesting to note that the model remains solvable if one adds to H the following interaction terms:

$$V_{\text{H}} = \sum v_{\ell m}\rho_{\ell\ell}\rho_{mm} + \tilde{v}_{\ell m}\tilde{\rho}_{\ell\ell}\tilde{\rho}_{mm} + w_{\ell m}\rho_{\ell\ell}\tilde{\rho}_{mm} \tag{27}$$

with $\tilde{v}_{\ell m}$, $v_{\ell m}$ and $w_{\ell m}$ arbitrary parameters: the eigenstates ψ in equation (23) remain the same, and the eigenvalues $\mathcal{E}$ are changed by adding $\sum_{i,j}\left(v_{m_i m_j} + \tilde{v}_{\ell_i\ell_j} + w_{m_i\ell_j}\right)$. I will come back to this in section 3.3.

2.5. Physical properties

I found all energy eigenstates and eigenstates of the QH model. However, this is only a first step to understand the physics described by this model: much work remains to be done. In this section I describe a few results and mention interesting open problems.

2.5.1. Correlations. The $|N\rangle$ in equation (19) are Slater states: they are eigenstates of the free Hamiltonian in equation (10) and describe fermions without any correlations. In the following section I will discuss the so-called Hartree-like models. These are exactly solvable models with interactions but such that the eigenstates are still Slater states: the interaction

only modifies the energy eigenvalues but the eigenstates remain uncorrelated. The QH model solved above is remarkable in that its eigenstates are particular linear combinations of Slater states. These eigenstates ψ still are characterized by N pairs of quantum numbers (ℓ_j, m_j), but they are complicated linear combinations of Slater states $|N\rangle$ where the ℓ_j and m_j are distributed over the fermions in many different ways. The particular distributions which yield eigenstates are labelled by partitions $[\lambda]$, and the energy of the state depends not only on the quantum numbers (ℓ_j, m_j) but also on $[\lambda]$. It is impossible, in general, to write these eigenstates ψ as Slater states. It would be very interesting to compute four-point Greens functions and thus understand the correlations quantitatively.

2.5.2. Ground-state and partition function. A first step in understanding the physical properties of the QH model is to determine the ground state at fixed fermions filling $\nu = N/L$ and in the thermodynamic limit $L \to \infty$. For that one needs to find the eigenstate ψ at fixed, large particle numbers N such that the corresponding eigenvalue $\mathcal{E}$ is minimal. This is a non-trivial problem whose solution I only know in one simple, special case.

The main difficulty in determining the ground state of the QH model from the solution given above is that I actually found too many eigenstates: for each set of N quantum numbers (ℓ_j, m_j) I constructed $N!$ eigenstates, but these can be linearly independent only if all ℓ_j and all m_j are distinct. For example, if all m_j in $|N\rangle$ equation (19) are the same, then $|N\rangle$ is only non-zero if all ℓ_j are distinct (Pauli principle). Moreover, in this case all permutations act trivially, $T_P|N\rangle = |N\rangle$, and thus there is only one non-zero eigenstate, namely $\psi = |N\rangle$, corresponding to $[\lambda] = [N]$ (the Young tableau with only one row). The opposite extreme is when all ℓ_j are the same. In this case all m_j need to be distinct, and the only non-zero eigenstate corresponds to $[\lambda] = [1^N]$ (the Young tableau with only one column). In general, degeneracies of the ℓ_j and m_j will reduce the multiplicities of the eigenstates given above. Many of these multiplicities are zero due to the Pauli principle, and to find the minimum energy eigenvalue with non-zero multiplicity can be difficult. This seemingly technical point has important physical implications: it can lead to a frustration in the system, and thus the ground state can change drastically as the model parameters are varied. Below I will illustrate this by discussing the ground state of the QH model as a function of g/B.

More generally one would like to compute the grand canonical partition function $\mathcal{Z} = \mathrm{Tr}\,\exp(-\beta H)$. The computation of $\mathcal{Z}$ seems like a highly non-trivial problem, but I believe it has a beautiful solution, at least for the case where $\tilde{E} = 0$: as already mentioned, one can represent $\mathcal{Z}$ as a matrix model with an external field E, and even though the resulting matrix model does not seem to be a 'standard' one it seems computable (it generalizes a matrix model recently solved in [13]). I now sketch how to represent $\mathcal{Z}$ as a matrix model. I start with a standard representation of $\mathcal{Z}$ as a fermion functional integral, i.e. an integral over τ-dependent Grassmann numbers where $0 \leqslant \tau \leqslant \beta$ is the Matsubara time (see e.g. [15]). One then introduces a Hubbard–Stratonovich field $Y(\tau)$, $0 \leqslant \tau \leqslant \beta$, which allows us to integrate out the fermions. The fermion integral yields a functional determinant $\det(E \otimes I + I \otimes [\partial_\tau - \mathrm{i}Y(\tau)])$. The field Y is a Hermitian $L \times L$ matrix-valued function which is periodic, $Y(0) = Y(\beta)$. Thus $\mathcal{Z}$ becomes a Hermitian matrix path integral. One then can change variables to $X = U(\tau)^{-1}[\mathrm{i}\partial_\tau + Y(\tau)]U(\tau)$ with $U(\tau)$ unitary matrices and periodic and X a Hermitian constant matrix (independent of τ). The integral over $U(\tau)$ can be done, and thus one can represent $\mathcal{Z}$ as an integral over the Hermitian matrices X in an external field E. I believe it is possible to evaluate this matrix integral in the large-L limit by generalizing the computation of a similar integral in [13].

2.5.3. Zero-temperature phase diagram of the QH model. I now discuss the ground state of the QH model

$$H = \sum_{m \geqslant 1} 4B(m-1)\rho_{mm} + g \sum_{\ell,m} : \rho_{\ell m}\rho_{m\ell} : \tag{28}$$

as a function of the coupling parameter g/B ($B > 0$, g arbitrary real, and I set $\mu = 1/2$ for convenience). As discussed, the energy eigenvalues are sums of two terms: the kinetic energy $\mathcal{E}_0 = \sum_{j=1}^{N} 4B(m_j - 1)$, and the energy $\mathcal{E}_{\rm corr} = 2g\gamma_{[\lambda]}$ due to the interaction. I consider the case $N < L$, i.e. the fermion filling factor $\nu = N/L$ is less than 1. Then the kinetic energy is minimal if all fermions have the same quantum number $m_j = 1$. In this case $|N\rangle$ in equation (19) is also an eigenstate of the interaction potential $V_{\rm QH}$, and the corresponding eigenvalue is $\mathcal{E}_{\rm corr} = gN(N-1)$. This obviously leads to a minimal total energy $\mathcal{E}_{[N]} = gN(N-1)$ if $g \leqslant 0$. However, for positive g it is preferable to have some fermions with $m_j > 1$: this increases the kinetic energy but allows us to decrease the correlation energy. In particular, for very large positive values of g the correlation energy will dominate. The minimum value $\mathcal{E}_{\rm corr} = -gN(N-1)$ of the correlation energy is for $[\lambda] = [1^N]$, but one needs all m_j to be different to get a non-zero such eigenstate. The minimum kinetic energy possible then is for $m_j = j$. This gives the total energy $\mathcal{E}_{[1^N]} = (2B - g)N(N-1)$, which is lower than $\mathcal{E}_{[N]}$ for $g > B$. One can show that the latter states are ground states for $g > 2B$. For general, fixed partition $[\lambda] = [\lambda_1, \ldots, \lambda_K]$, a state of lowest energy is

$$\Psi_{[\lambda]} = c^\dagger_{1,1}c^\dagger_{2,1}\cdots c^\dagger_{\lambda_1,1}c^\dagger_{1,2}c^\dagger_{2,2}\cdots c^\dagger_{\lambda_2,2}\cdots c^\dagger_{1,K}c^\dagger_{2,K}\cdots c^\dagger_{\lambda_K,K}\Omega \tag{29}$$

and the corresponding energy eigenvalue is

$$\mathcal{E}_{[\lambda]} = \sum_{j=1}^{K} 4B\lambda_j(j-1) + g(\lambda_j(\lambda_j+1) - 2j\lambda_j). \tag{30}$$

The ground states in the intermediate regime $0 < g < 2B$ can be found by minimizing $\mathcal{E}_{[\lambda]}$ with the constraint $\sum_j \lambda_j = N$. One finds that they are given by partitions

$$\lambda_j = \text{non-negative integer close to } a - bj \qquad 0 \leqslant j \leqslant a/b \tag{31}$$

with $b = (2B - g)/g > 0$ and $a = \sqrt{2Nb}$ (the corresponding Young tableau approximates a right triangle with area N and slope b). One thus finds that the QH model has a non-trivial zero-temperature phase diagram with zero-temperature phase transitions at $g = 0$ and $g = 2B$. The ground states are Slater states for $g < 0$ and $g > 2B$ but are correlated states for $0 < g < 2B$. All ground states are highly degenerated for $1 \ll N \ll L$, and the degeneracy (entropy) varies significantly in the region $0 < g < 2B$.

3. Interacting fermion systems

In this section I discuss several well-known exactly solvable models of interacting fermions which have played an important role in condensed matter physics. This discussion suggests an interesting physical interpretation of the QH model discussed in the previous section. This interpretation will lead me to propose a general strategy to obtain 'interesting' simplified models for 2D correlated fermions.

The general class of interacting fermion models which I consider describe fermions with one-particle states labelled by a large but finite number of different quasi-momenta $\mathbf{k}$ and a spin index α assuming at most two different values ('at most' since I also discuss models for spinless fermions). The models are defined by an Hamiltonian $H = H_0 + V$ with the free part

$$H_0 = \sum_{\mathbf{k},\alpha} (E_{\mathbf{k}} - \mu) c^\dagger_{\mathbf{k},\alpha} c_{\mathbf{k},\alpha} \tag{32}$$

where $E_{\mathbf{k}}$ is the dispersion relation and μ the chemical potential. I assume that the interaction potential is of the following form:

$$V = \sum{}' \sum_{\alpha,\beta} v_{\mathbf{k}_1\ldots\mathbf{k}_4} : c^\dagger_{\mathbf{k}_1,\alpha} c_{\mathbf{k}_2,\alpha} c^\dagger_{\mathbf{k}_3,\beta} c_{\mathbf{k}_4,\beta} : \tag{33}$$

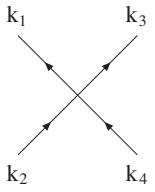
where $v_{\mathbf{k}_1\ldots\mathbf{k}_4}$ is the interaction vertex in Fourier space. The $c^{(\dagger)}_{\mathbf{k},\alpha}$ are fermion operators obeying the usual anticommutation relations

$$\left\{c^\dagger_{\mathbf{k},\alpha}, c_{\mathbf{k}',\beta}\right\} = \delta_{\mathbf{k},\mathbf{k}'}\delta_{\alpha,\beta} \tag{34}$$

etc. The model is defined on the fermion Fock space with a vacuum Ω such that $c_{\mathbf{k},\alpha}\Omega = 0$ for all α and $\mathbf{k}$. The primed sum is over all $\mathbf{k}_1, \ldots, \mathbf{k}_4$ restricted by momentum conservation. In simple cases the latter amounts to $\mathbf{k}_1 - \mathbf{k}_2 + \mathbf{k}_3 - \mathbf{k}_4 = \mathbf{0}$, but if the system is defined on a spatial lattice with finite lattice constant also umklapp processes are allowed and '$\mathbf{0}$' above can be replaced by any reciprocal lattice vector.

In the following discussion I will symbolically represent the different interaction terms in (33) as follows:

$$: c^\dagger_{\mathbf{k}_1,\alpha} c_{\mathbf{k}_2,\alpha} c^\dagger_{\mathbf{k}_3,\beta} c_{\mathbf{k}_4,\beta} : = \tag{35}$$



where spin conservation is understood horizontally. This term represents the process where two fermions with momenta $\mathbf{k}_2$ and $\mathbf{k}_4$ are scattered and thereby obtain new momenta $\mathbf{k}_1$ and $\mathbf{k}_3$, respectively. The Hubbard model on a finite 2D $L \times L$ lattice corresponds to the special case where[4]

$$\mathbf{k} = (k_x, k_y), \qquad k_{x,y} = (2\pi/L) \times \text{integer}, \qquad -\pi \leqslant k_{x,y} < \pi, \tag{36}$$

$\alpha = \uparrow, \downarrow$ and

$$E_{\mathbf{k}} = -2t(\cos k_x + \cos k_y), \qquad v_{\mathbf{k}_1\ldots\mathbf{k}_4} = \frac{U}{2L^2} \sum_{\mathbf{K}} \delta_{\mathbf{k}_1-\mathbf{k}_2+\mathbf{k}_3-\mathbf{k}_4,\mathbf{K}}, \tag{37}$$

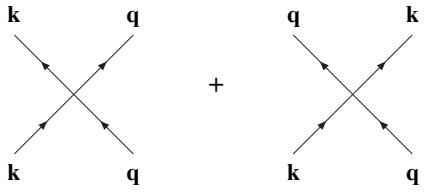
with t and $U > 0$ the usual Hubbard constants; the sum is over all reciprocal lattice vectors $\mathbf{K} = (K_x, K_y)$ with $K_{x,y} = 2\pi \times$ integer. Thus in the Hubbard interaction potential all scattering terms occur and have equal weight. The number of distinct one-particle states in this case is $2L^2$, and the fermion Fock space is 2^{2L^2} dimensional.

I now discuss various simplified models which can be obtained from a models of the type given above. The method I use is by truncation, leaving only particular interaction terms. This truncation method might seem ad hoc, and I therefore stress that it only should be regarded as a simple way to find the *structure* of 'interesting' Hamiltonians. Systematic derivations of these Hamiltonians should be based on renormalization group methods (see e.g. [16] and references therein).

[4] To see that this is identical with the standard definition of the 2D Hubbard model (see e.g. [2]) use $n_\uparrow n_\downarrow = : (n_\uparrow + n_\downarrow)^2 : /2$ and perform a lattice Fourier transform.

3.1. Hartree–Fock models

As is well known, not all interaction terms in an interacting fermion model are equally important. For example, the terms where the particle momenta are conserved, $\mathbf{k}_1 = \mathbf{k}_2$ and $\mathbf{k}_3 = \mathbf{k}_4$, or exchanged, $\mathbf{k}_1 = \mathbf{k}_4$ and $\mathbf{k}_3 = \mathbf{k}_2$, are known to be of particular importance. I will refer to them as as Hartree and Fock terms, respectively. They can be represented symbolically as follows:



Hartree terms Fock terms (38)

The simplest example of an exactly solvable model is obtained by truncating the interaction potential in (33) and keeping only the Hartree terms,

$$V_{\mathrm{H}} = \sum_{\mathbf{k},\mathbf{q},\alpha,\beta} W_{\mathbf{k},\mathbf{q}} : \hat{n}_{\mathbf{k},\alpha} \hat{n}_{\mathbf{q},\beta} : \tag{39}$$

where I introduced the fermion number operators

$$\hat{n}_{\mathbf{k},\alpha} \equiv c^{\dagger}_{\mathbf{k},\alpha} c_{\mathbf{k},\alpha}. \tag{40}$$

These operators all commute, and thus the Hartree Hamiltonians $H = H_0 + V_{\mathrm{H}}$ are sums of commuting terms and can be easily diagonalized. Note that one obtains such a model also if one also keeps parts of the Fock terms, namely those with $\alpha = \beta$: this adds to the interaction potential the terms

$$V_{\mathrm{F}} = -\sum_{\mathbf{k},\mathbf{q},\alpha} W_{\mathbf{k},\mathbf{q}} : \hat{n}_{\mathbf{k},\alpha} \hat{n}_{\mathbf{q},\alpha} : \tag{41}$$

and all I say below can be easily generalized to Hartree-type models with Hamiltonians $H = H_0 + V_{\mathrm{H}} + V_{\mathrm{F}}$.

I naively described the procedure yielding the simplified Hamiltonian $H = H_0 + V_{\mathrm{H}}$ from the one in equations (32) and (33) as truncation dropping all but certain particular terms in the interaction potential. This suggests simple formulae of the parameters $W_{\mathbf{k},\mathbf{q}}$ in terms of the $v_{\mathbf{k}_1,\ldots,\mathbf{k}_4}$. However, a systematic derivation should be based on some renormalization procedure in which the model parameters are renormalized and all but certain interaction terms become irrelevant. Thus this truncation should also be accompanied by some change of model parameters. One thus should interpret the Hamiltonian $H = H_0 + V_{\mathrm{H}}$ as a phenomenological model with model parameters to be fixed by experimental data. Similar remarks apply to all other truncated models discussed below.

The Hartree Hamiltonian $H = H_0 + V_{\mathrm{H}}$ is exactly solvable: as can be checked easily, its exact energy eigenstates are given by

$$\Psi_{\mathbf{n}} = \prod_{\mathbf{k},\alpha} \left(c^{\dagger}_{\mathbf{k},\alpha}\right)^{n_{\mathbf{k},\alpha}} \Omega \tag{42}$$

and are labelled by the occupation numbers $\mathbf{n} = \{n_{\mathbf{k},\alpha}\}$, $n_{\mathbf{k},\alpha} = 0, 1$. The corresponding energy eigenvalues are

$$\mathcal{E}_{\mathbf{n}} = \sum_{\mathbf{k},\alpha} (E_{\mathbf{k}} - \mu) n_{\mathbf{k},\alpha} + \sum_{\mathbf{k}\neq\mathbf{q},\alpha,\beta} W_{\mathbf{k},\mathbf{q}} n_{\mathbf{k},\alpha} n_{\mathbf{q},\beta}. \tag{43}$$

This exact solution makes the analysis of this model much simpler as compared to a general interacting fermion model in equations (32)–(34), but it important to note that this solution still is quite far from understanding the physics of this model: to find the ground state of the Hamiltonian in (39) at fixed particle number N one needs to minimize the functional in equation (43) over all configurations $\mathbf{n}$ such that $\sum_{\mathbf{k},\alpha} n_{\mathbf{k},\alpha} = N$, and even though the exact solutions yields the following formula for the partition function:

$$\mathcal{Z} = \sum_{\mathbf{n}} \mathrm{e}^{-\beta \mathcal{E}_{\mathbf{n}}} \tag{44}$$

its computation still is a non-trivial task (it is equivalent to solving an Ising-type model[5]). It is interesting to note that such an analysis of a Hartree-type model appears to be very similar to Landau's Fermi liquid theory [17].

An important feature of the Hartree-type models is that their eigenstates are identical with the ones of the corresponding non-interacting model: the interaction only affects the energy eigenvalues, but the eigenstates remain Slater states without any fermion correlations. A more complicated model taking into account some correlations is obtained if one keeps, in addition to Hartree terms, all Fock terms (see (38) above). This yields an interaction potential of the following type:

$$V_{\mathrm{HF}} = \sum_{\mathbf{k},\mathbf{q}} : W_{\mathbf{k},\mathbf{q}} \hat{n}_{\mathbf{k}} \hat{n}_{\mathbf{q}} + J_{\mathbf{k},\mathbf{q}} \hat{\mathbf{S}}_{\mathbf{k}} \cdot \hat{\mathbf{S}}_{\mathbf{q}} : \tag{45}$$

with the usual spin- and charge density operators $\hat{\mathbf{S}} = (\hat{S}^1, \hat{S}^2, \hat{S}^3)$ and $\hat{n}$,

$$\hat{\mathbf{S}}_{\mathbf{k}} = \sum_{\alpha,\beta} c^\dagger_{\mathbf{k},\alpha} (\boldsymbol{\sigma})_{\alpha\beta} c_{\mathbf{k},\beta} \qquad \text{and} \qquad \hat{n}_{\mathbf{k}} = \sum_{\alpha} c^\dagger_{\mathbf{k},\alpha} c_{\mathbf{k},\alpha} \tag{46}$$

and $\boldsymbol{\sigma} = (\sigma^1, \sigma^2, \sigma^3)$ the Pauli spin matrices as usual.

The Hamiltonian $H = H_0 + V_{\mathrm{HF}}$ is no longer the sum of mutually commuting terms. Its eigenfunctions thus are more complicated. To see that it is convenient to write the Slater states in equation (42) as follows:

$$|\alpha_1, \ldots, \alpha_N\rangle = c^\dagger_{\mathbf{k}_1,\alpha_1} \cdots c^\dagger_{\mathbf{k}_N,\alpha_N} \Omega. \tag{47}$$

The action of $H = H_0 + V_{\mathrm{HF}}$ on these states is easily computed,

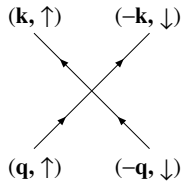
$$H|\alpha_1, \ldots, \alpha_N\rangle = \left(\mathcal{E}_0 + \sum_{j \neq k} J(\mathbf{k}_j, \mathbf{k}_k) \hat{\boldsymbol{\sigma}}_j \cdot \hat{\boldsymbol{\sigma}}_k \right) |\alpha_1, \ldots, \alpha_N\rangle \tag{48}$$

with $\mathcal{E}_0 = \sum_j \left(E_{\mathbf{k}_j} - \mu \right) + \sum_{j \neq k} W_{\mathbf{k}_j,\mathbf{k}_j}$ the Hartree-model eigenvalues and the $\hat{\boldsymbol{\sigma}}_j$ Heisenberg spin operators acting on the spin α_j-index of the fermions as usual. To find the eigenvalues of the Hartree–Fock model one thus needs to diagonalize the spin Hamiltonians $\sum_{j \neq k} J(\mathbf{k}_j, \mathbf{k}_k) \hat{\boldsymbol{\sigma}}_j \cdot \hat{\boldsymbol{\sigma}}_k$. The latter can be interpreted as 1D Heisenberg spin systems with long-range interactions. In the case where $J(\mathbf{k}, \mathbf{q}) = J$ is constant, the spin Hamiltonian reduces to $J \left(\sum_{j=1}^N \hat{\boldsymbol{\sigma}}_k \right)^2 + \text{const}$, and its eigenvalues and eigenstates can be determined using group theory (i.e. decomposing the direct sum of N spin-1/2 representations of the Lie algebra of SU(2) into irreps).

[5] I thank M Salmhofer for this observation.

3.2. BCS-type models

These models underlie the theory of of superconductivity. They can formally be obtained from a general interacting fermion model in equations (32)–(34) by dropping all but the following terms in the interaction potential:



BCS terms (49)

The resulting Hamiltonian is $H = H_0 + V_{\mathrm{BCS}}$ with

$$V_{\mathrm{BCS}} = \sum_{\mathbf{k},\mathbf{q}} g_{\mathbf{k},\mathbf{q}} \Delta^+_{\mathbf{k}} \Delta^-_{\mathbf{q}} \tag{50}$$

where

$$\Delta^+_{\mathbf{k}} = c^\dagger_{\mathbf{k},\uparrow} c^\dagger_{-\mathbf{k},\downarrow} \qquad \Delta^-_{\mathbf{q}} = c_{-\mathbf{q},\downarrow} c_{\mathbf{q},\uparrow}. \tag{51}$$

This Hamiltonian is exactly solvable for parity invariant systems where $E_{\mathbf{k}} = E_{-\mathbf{k}}$. One way to see this is to write $H_0 = \sum_{\mathbf{k}} (E_{\mathbf{k}} - \mu)\tilde{S}_{\mathbf{k}}$ (up to an irrelevant additive constraint) where

$$\tilde{S}_{\mathbf{k}} = c^\dagger_{\mathbf{k},\uparrow} c_{\mathbf{k},\uparrow} - c_{-\mathbf{k},\downarrow} c^\dagger_{-\mathbf{k},\downarrow}. \tag{52}$$

One can check that the operators $\Delta^\pm$ and $\tilde{S}$ represent the Lie algebra of SU(2),

$$\left[\Delta^+_{\mathbf{k}}, \Delta^-_{\mathbf{q}}\right] = \delta_{\mathbf{k},\mathbf{q}} \tilde{S}_{\mathbf{k}}, \qquad \left[\tilde{S}_{\mathbf{k}}, \Delta^\pm_q\right] = \pm\delta_{\mathbf{k},\mathbf{q}} \Delta^\pm_{\mathbf{k}}. \tag{53}$$

Thus the BCS Hamiltonian is a quantum SU(2) spin-type model of similar to the Hartree–Fock model described above. One can make this relation more specific by introducing the following Bogoliubov transformation:

$$\tilde{c}_{\mathbf{k},\uparrow} \equiv c_{\mathbf{k},\uparrow}, \qquad \tilde{c}_{\mathbf{k},\downarrow} \equiv c^\dagger_{-\mathbf{k},\downarrow} \tag{54}$$

leaving the canonical anticommutation relations invariant. Then $\Delta^\pm_{\mathbf{k}} = \left(\tilde{S}^1_{\mathbf{k}} \pm \mathrm{i}\tilde{S}^2_{\mathbf{k}}\right)/2$ and $\tilde{S}_{\mathbf{k}} = \tilde{S}^3_{\mathbf{k}}$ with $\tilde{\mathbf{S}}_{\mathbf{k}} = \tilde{c}^\dagger_{\mathbf{k}} \boldsymbol{\sigma} \tilde{c}_{\mathbf{k}}$ the spin operators associated with these new fermion operators. The BCS Hamiltonian can then be written as

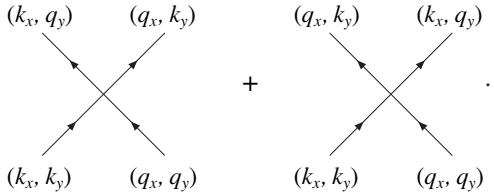
$$H = \sum_{\mathbf{k}} (E_{\mathbf{k}} - \mu)\tilde{S}^3_{\mathbf{k}} + \sum_{\mathbf{k},\mathbf{q}} g_{\mathbf{k},\mathbf{q}} \left(\tilde{S}^1_{\mathbf{k}} \tilde{S}^1_{\mathbf{q}} + \tilde{S}^2_{\mathbf{k}} \tilde{S}^2_{\mathbf{q}}\right) \tag{55}$$

which formally is identical with a xy Heisenberg spin model in a transverse magnetic field. This model can therefore be treated in a similar manner as the Hartree–Fock model discussed above. The simplest case of constant coupling, $g_{\mathbf{k},\mathbf{q}} = g$, underlies the BCS theory of superconductivity [5] and was solved exactly in [6].

3.3. A different interpretation of the QH model

My discussion above suggests the following interpretation of the interaction potential of the QH model: one can regard the quantum numbers ℓ and m as the x- and y-components of a 2D pseudo momentum, $\ell = k_x$ and $m = k_y$. One then can interpret V_{QH} in equation (12) as mixed

scattering terms which are Hartree-like in the x-component and Fock-like in the y-component of the momenta or vice versa, i.e.



(56)

As discussed, one can exactly solve the generalized QH model $H = H_0 + V_{\rm QH} + V_{\rm H}$ defined in equations (10), (12) and (27). Interpreting $E_m + \tilde{E}_\ell \equiv E_{\mathbf{k}}$ as dispersion relation, one can regard this as simplified model obtained by truncating a spin-less variant of a model defined in equations (32) and (33), keeping not only Hartree and Fock terms but also the mixed terms just described. One main point of this paper is that it is possible to keep these mixed terms and still have an exactly solvable model.

As is well known, spatial correlations are important in 2D, and the simplified models models including only the Hartree and Fock terms (38) are not adequate. The main suggestion is to use the generalized models in 2D where, in addition to the Hartree–Fock terms (38), also the mixed terms (56) are included.

It is important to note that, in my present interpretation of the QH model, interchanging $k_x \leftrightarrow k_y$ amounts to changing $E_\ell \leftrightarrow \tilde{E}_\ell$ and $g \leftrightarrow -g$; (see equations (13)–(15)). Thus, for spin-less fermions, one can have a non-trivial correlation interaction potential $V_{\rm QH}$ only if parity invariance is broken. As discussed below, this no longer is the case for fermions with spin.

3.4. Generalized Hartree model

I now define a parity invariant variant of the QH model including spin. It corresponds to a generalization of a Hartree model $H_0 + V_{\rm H}$ defined in equations (32) and (39). I restrict myself to 2D dispersion relation of the following form, $E_{\mathbf{k}} = e_{k_x} + e_{k_y}$, and assume that the Hartree coupling can be written as $W_{\mathbf{k},\mathbf{q}} = v_{k_x q_x} + v_{k_y q_y} + w_{k_x q_y} + w_{q_x k_y}$. Introducing the operators

$$\rho_{k_y q_y} = \sum_{k_x,\alpha} c^\dagger_{k_x k_y \alpha} c_{k_x q_y \alpha}, \qquad \tilde{\rho}_{k_y q_y} = \sum_{k_y,\alpha} c^\dagger_{k_x k_y \alpha} c_{q_x k_y \alpha} \tag{57}$$

one can write

$$H_0 = \sum_k e_k(\rho_{kk} + \tilde{\rho}_{kk} - \mu) \tag{58}$$

and

$$V_{\rm H} = \sum_{k,q} v_{k,q}(\rho_{kk}\rho_{qq} + \tilde{\rho}_{kk}\tilde{\rho}_{qq}) + w_{k,q}(\rho_{kk}\tilde{\rho}_{qq} + \tilde{\rho}_{kk}\rho_{qq}). \tag{59}$$

I now propose the model $H = H_0 + V_{\rm H} + V_{\rm mixed}$ with the following mixed interaction potential:

$$V_{\rm mixed} = \sum_{k,q} g : (\rho_{kq}\rho_{qk} + \tilde{\rho}_{kq}\tilde{\rho}_{qk} :). \tag{60}$$

This model is obviously parity invariant, and due to the presence of spin the interaction potential $V_{\rm mixed}$ is non-trivial. It is straightforward to generalize the discussion in 2.3 and 2.4. One finds that this model also has a dynamical symmetry $\mathrm{gl}_\infty \oplus \mathrm{gl}_\infty$, and explicit formulae for its eigenstates and eigenvalues can be obtained as well. It would be interesting to explore the physics of this model in more details.

3.5. On the 2D Hubbard model

To illustrate the flexibility of the truncation procedure advocated above, I now discuss various simplified models for the 2D Hubbard model. I also show that the exact solution of the Hartree-like models for the 2D Hubbard model reproduce mean field theory, as claimed in the introduction.

3.5.1. Simplified models. For the 2D Hubbard model the Hartree-like model $H_1 = H_0 + V_H + V_F$ defined in equations (32), (39) and (41) is given by

$$H_0 = \sum_{\mathbf{k}} E_{\mathbf{k}} \hat{n}_{\mathbf{k}}, \qquad V_H + V_F = \frac{U}{L^2} \hat{n}_\uparrow \hat{n}_\downarrow \tag{61}$$

with $E_{\mathbf{k}}$ in (37) and $\hat{n}_\alpha = \sum_{\mathbf{k}} \hat{n}_{\mathbf{k},\alpha}$. As I show below, this model is equivalent to mean-field theory (Hartree–Fock theory) restricted to ferromagnetic (F) states. It misses an important physical property of the 2D Hubbard model which, at half filling, has anti-ferromagnetic (AF) rather than F order (see e.g. [2]). It thus is interesting to note that there is also an exactly solvable model accounting for AF: it it obtained by including Hartree-like terms where the momenta are not fully conserved but changed by the AF vector $\mathbf{Q} = (\pi, \pi)$, $\mathbf{k}_1 = \mathbf{k}_2 + \mathbf{Q}$ and $\mathbf{k}_3 = \mathbf{k}_4 + \mathbf{Q}$, and similar Fock-like terms[6]. These yields the following interaction potential:

$$V_H^{\mathbf{Q}} + V_F^{\mathbf{Q}} = \frac{U}{L^2} \hat{n}_\uparrow^{\mathbf{Q}} \hat{n}_\downarrow^{\mathbf{Q}}, \qquad n_\alpha^{\mathbf{Q}} = \sum_{\mathbf{k}} c^\dagger_{\mathbf{k}+\mathbf{Q},\alpha} c_{\mathbf{k},\alpha}. \tag{62}$$

The model $H_2 = H_1 + V_H^{\mathbf{Q}} + V_F^{\mathbf{Q}}$ can be solved exactly, and its solution yields Hartree–Fock theory for the 2D Hubbard model allowing for F, AF, and charge-density waves.

The corresponding mixed terms are

$$V_{\text{mixed}} = \frac{U}{L^2} \sum_{k,q} : \rho_{kq\uparrow} \rho_{qk\uparrow} + \tilde{\rho}_{kq\uparrow} \tilde{\rho}_{qk\uparrow} : \tag{63}$$

with ρ_{kq} and $\tilde{\rho}_{kq}$ in (57), and similarly $V_{\text{mixed}}^{\mathbf{Q}}$. It would be interesting to determine the phase diagram of the model $H_3 = H_2 + V_{\text{mixed}} + V_{\text{mixed}}^{\mathbf{Q}}$. I hope to come back to that in future work.

3.5.2. Relation to mean-field theory. I now show how to compute the partition functions of the Hartree-like models above, and that this reproduces standard mean-field theory. For simplicity I restrict myself to the Hartree-like model allowing for AF only,

$$H = H_0 + \frac{U}{2L^2} (\hat{n}_\uparrow + \hat{n}_\downarrow)^2 + \frac{U}{2L^2} \left(\hat{n}_\uparrow^{\mathbf{Q}} - \hat{n}_\downarrow^{\mathbf{Q}}\right)^2 \tag{64}$$

(one can ignore normal ordering here since $\hat{n}_\sigma^2 = \hat{n}_\sigma$, and thus normal ordering only amounts to a shift in the chemical potential). It is easy to extend my argument to the other models discussed above. I use the following identity (Hubbard–Stratonovich transformation):

$$\mathrm{e}^{-\beta H} = \text{const} \int_{\mathbb{R}^2} \mathrm{d}r\, \mathrm{d}s \exp\left(-\beta\left(L^2(r^2+s^2)/U + H_0 + s\left(\hat{n}_\uparrow^{\mathbf{Q}} - \hat{n}_\downarrow^{\mathbf{Q}}\right) + \mathrm{i}r(\hat{n}_\uparrow + \hat{n}_\downarrow)\right)\right)$$

which linearizes the interaction at the cost of introducing two integrations. With that one can compute the partition function (I drop the irrelevant constant),

$$\mathcal{Z} = \int_{\mathbb{R}^2} \mathrm{d}r\, \mathrm{d}s \exp(-\beta L^2[(r^2+s^2)/U - F(r,s)]) \tag{65}$$

[6] These are umklapp processes included in the Hubbard interaction potential in equation (37).

with $\mathrm{e}^{-\beta L^2 F}$ the partition function of non-interacting fermions coupled to constant external fields r and s. The latter can be easily computed, and one obtains

$$F = \sum_{\alpha=\pm} \frac{1}{L^2} \sum_{\mathbf{k}} \log(1 + \mathrm{e}^{-\beta\varepsilon_{\mathbf{k},\alpha}}) \tag{66}$$

with

$$\varepsilon_{\mathbf{k},\pm} = (E_{\mathbf{k}} + E_{\mathbf{k}+\mathbf{Q}})/2 - \mu \pm \sqrt{s^2 + (E_{\mathbf{k}} - E_{\mathbf{k}+\mathbf{Q}})^2/4} \tag{67}$$

the fermion bands in an AF background. Note that $L^{-2}\sum_{\mathbf{k}}$ is a Riemann sum converging to an integral in the thermodynamic limit $L \to \infty$. In this limit a saddle point evaluation becomes exact. The saddle point equations one thus obtains are identical with Hartree–Fock theory for the Hubbard model restricted to AF states (see e.g. [2]).

It is interesting to note that equation (65) was obtained by a different method in [18]. In this work we proposed it as a useful alternative formulation of mean-field theory for the 2D Hubbard model. Superficially this seems equivalent to the 'standard' formulation which only uses the saddle point equations resulting from that integral, but it is in fact more general since it allows for the possibilities of having degenerate saddle points. We found that degenerate saddle points occur in a large part of the parameter regime: if one fixes the fermion density close but away from half filling one typically needs to adjust the chemical potential μ such that the integral is dominated by *two* distinct saddle points at the same time, one describing AF order, and another describing no order at all [18]. These mixed phases are missed if one only looks at the saddle point equations: the mean-field phase diagram of the 2D Hubbard model thus obtained is much richer than generally believed (see e.g. [2] and references therein). Mixed phases in simple mean-field theory are typical in Hubbard-like models. I feel this is a basic, important property of these models which deserves to be more widely known. Anyway, the present paper gives models for which the mixed phases are exact.

4. Conclusions

In this paper I advocated the following strategy for treating difficult many-body Hamiltonians for interacting fermions: rather than solving such model in some approximation (such mean-field theory), truncate it so as to obtain a simplified model which can be solved exactly. I recalled examples from condensed matter physics a long time ago where this strategy was used successfully to solve important problems such as superconductivity or magnetism. I also discussed a many-body model for 2D quantum Hall systems proposed and solved in [1]. I then argued that this model is the simplest example in a large class of similar models which should be useful for shedding light on an interesting problem in modern condensed matter physics: to understand 2D correlated fermion systems. As indicated, the partition function of these generalized models can be represented as matrix models. The computations of the latter are interesting problems for the future.

Many of the results reported here can be adapted straightforwardly to boson systems.

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